

Quantum operations that cannot be implemented using a small mixed environment.

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Abstract

To implement any quantum operation (a.k.a. “superoperator” or “CP map”) on a d -dimensional quantum system, it is enough to apply a suitable overall unitary transformation to the system and a d^2 -dimensional environment which is initialized in a fixed pure state. It has been suggested that a d -dimensional environment might be enough if we could initialize the environment in a mixed state of our choosing. In this note we show with elementary means that certain explicit quantum operations cannot be realized in this way. Our counterexamples map some pure states to pure states, giving strong and easily manageable conditions on the overall unitary transformation. Everything works in the more general setting of quantum operations from d -dimensional to d' -dimensional spaces, so we place our counterexamples within this more general framework.

1 Quantum operations

Quantum operations (see e.g. [3]) are also known as “superoperators”, “super-scattering operators” or “completely positive maps” (“CP maps”). They can be viewed as a generalization of unitary transformations and are the most general transformations that can be applied to a quantum system in an unknown (possibly mixed) state. More precisely, quantum operations are the most general transformations that can be implemented deterministically, thus excluding operations which only succeed with a certain probability, like those depending on a measurement outcome. Under a quantum operation pure states are frequently mapped to mixed states.

All quantum operations on a d -dimensional system can be implemented as the partial trace of a unitary operator acting on the system together with an auxiliary system (the “environment”). The question is how small an environment suffices to implement all possible quantum operations on a d -dimensional

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system. The answer is easy, and has long been known, for the case in which the environment is initialized in a pure state. The answer is more interesting when the environment can be initialized in a mixed state.

2 Initializing the environment in a pure state

The environment is initialized in a standard pure state (let's call it $|0\rangle$). We apply an overall unitary transformation to the system plus environment (which typically involves interaction between the two). Finally we are only interested in the (now generally mixed) state of the system alone, thus we “trace over” the environment. Decoherence is a typical example of such a process. It is easy to see that quantum operations act linearly on density matrices.

We will think of quantum operations as given in the above way, as overall unitary transformations on a system together with an environment. For example, a quantum operation on a single qubit system can be given by the action of an overall unitary transformation on basis vectors for the space consisting of the system together with the environment. As the environment starts out in the state $|0\rangle$, we need only to give the images of the vectors $|i\rangle|0\rangle$ (where the first vector corresponds to the system and the second to the environment):

$$\begin{aligned} |0\rangle|0\rangle &\rightarrow |0\rangle|\psi_{00}\rangle + |1\rangle|\psi_{01}\rangle \\ |1\rangle|0\rangle &\rightarrow |0\rangle|\psi_{10}\rangle + |1\rangle|\psi_{11}\rangle \end{aligned}$$

where the environment vectors $|\psi_{ij}\rangle$ have to be such that the two right hand sides are orthonormal.

The action of the quantum operation on (density) operators is obtained by first lifting the above overall unitary transformation to the level of density operators and then tracing over the environment (using $\text{tr}|\varphi\rangle\langle\psi| = \langle\psi|\varphi\rangle$). Thus

$$|i\rangle|0\rangle\langle 0|\langle i'| \xrightarrow{U} \sum_{jj'} |j\rangle|\psi_{ij}\rangle\langle\psi_{i'j'}|\langle j'| \xrightarrow{\text{tr}_{env}} \sum_{jj'} |j\rangle\langle j'| \langle\psi_{i'j'}|\psi_{ij}\rangle.$$

Since only the inner product of the $|\psi_{ij}\rangle$ matter, we may assume that $|0\rangle = |\psi_{00}\rangle$. Thus we see that any quantum operation on a single qubit system can be implemented with a 4-dimensional environment, as the $|\psi_{ij}\rangle$ span at most a 4-dimensional space. For a d -dimensional system we have d^2 environment vectors $|\psi_{ij}\rangle$, so an environment of that dimension will always do. It is also not hard to show that in this setting some quantum operations really need an environment of this size.

2.1 Initializing the environment in a mixed state

While it does not matter in which pure state we initialize the environment, initializing in a mixed state rather than a pure state does matter. A mixed state is not unitarily equivalent to a pure state, rather the unitary equivalence is characterized by the eigenvalues p_i of the density matrix of the state. The

question at hand is whether we could implement any quantum operation with an environment smaller than d^2 if, for each map we want to implement, we are allowed to initialize the environment in a mixed state of our choosing.

Without loss of generality we can assume the initial state of the environment to be of the form $\rho_{env,initial} = \sum_i p_i |i\rangle\langle i|$. Because the environment now is smaller, the overall unitary transformation has fewer parameters, but more of them matter; it is no longer sufficient to know only the action on the subspace in which the environment is in state $|0\rangle$.

A parameter count (including the choice of the p_i) suggests that an environment of the same dimension as the system might just be enough to implement any quantum operation were we allowed to choose the initial mixed state of the environment. Thus the conjecture in [2] (see eq.(2)). We give a more general version of the parameter count in section 2.2.1.

It has been shown [1] for the qubit case ($d=2$) that this conjecture is not true; there are quantum operations which cannot be implemented with a mixed single qubit environment. The proof in [1] was rather “brute force”, as it used computer algebra to show that a system of equations does not have a solution. Here we give a simple proof using only elementary means that also generalizes to arbitrary dimensions.

2.2 General framework

In general we consider maps between systems of (possibly) different dimensions d and d' . We embed the “initial” d -dimensional system in a larger system, apply a unitary operator, and then take the partial trace over everything except the d' -dimensional “final” system. Thus physically we start with a system in some (generally mixed) state and end with a different system in some state. Note that the dimension can increase or decrease.

When the larger system contains a d' -dimensional system and an auxiliary environment initialized in pure states (see Figure 1), the same argument as in Section 2 shows that besides the d -dimensional initial system and the d' -dimensional final system, a d -dimensional environment is enough (and in general necessary) to realize general quantum operations between the two systems.

Were we to initialize the “final” system in a mixed state, a parameter count (misleadingly) suggests that the d -dimensional auxiliary environment could simply be left away. We place our counterexamples in this general framework of maps from a d -dimensional “initial” system to a d' -dimensional final (or “target”) system initialized in a mixed state of our choosing. An overall unitary transformation is applied to the initial plus final system, and then, as we are only interested in the state of the final system, we “trace over” the initial one. (See Figure 2.)

Note that the question we were initially interested in, of whether a d -dimensional environment that can be initialized in a mixed state is sufficient to implement any quantum operation on a d -dimensional system, can be answered in this general framework by looking at initial and final systems of the same dimension d and then swapping them at the end.

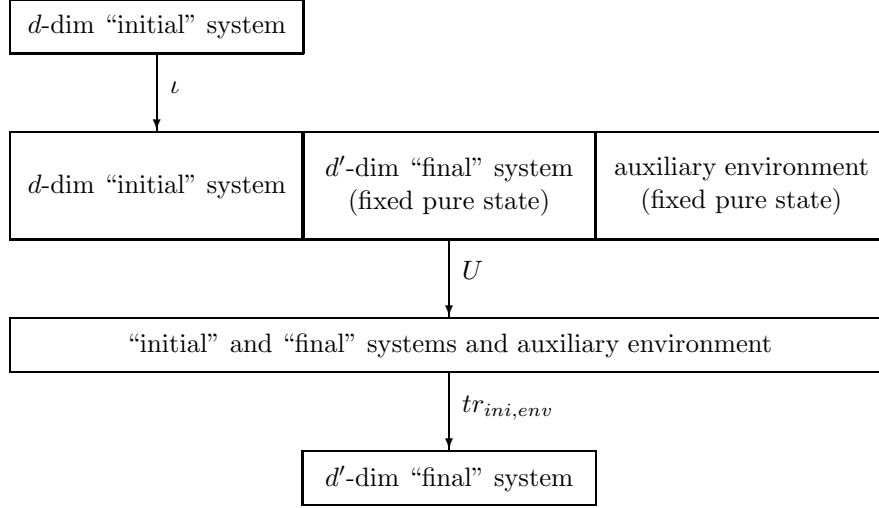


Figure 1: Map from d -dimensional "initial" system to d' -dimensional "final" system induced by unitary transformation U acting on the two systems and an auxiliary environment. The "final" system and the auxiliary environment are initialized in fixed pure states.

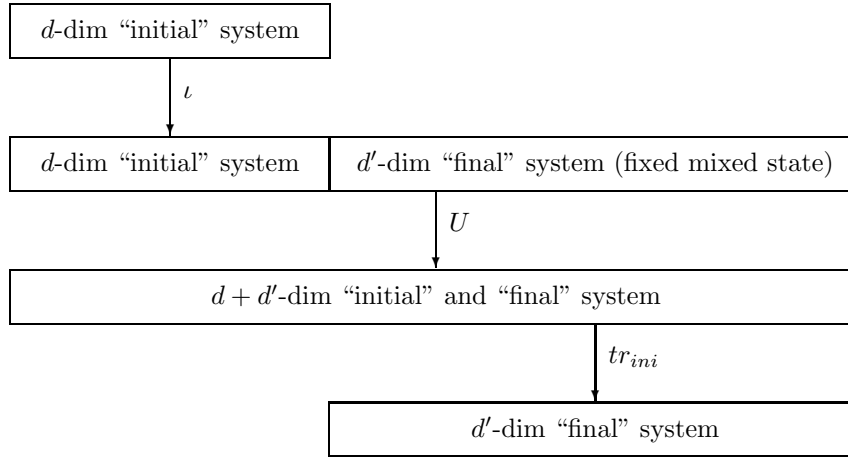


Figure 2: General framework: map from d -dimensional "initial" system to d' -dimensional "final" system induced by unitary transformation U . The "final" system is initialized in a fixed mixed state.

2.2.1 A sketch of the parameter count

Here is a sketch of the parameter count that leads to the wrong conjecture. The d' -dimensional mixed state in the standard form is given by $d' - 1$ (real) parameters. Note that it is invariant under diagonal (special) unitary transformations (“rephasings”). The overall (special) unitary transformation has $d^2 d'^2 - 1$ parameters. Finally we have to take into account that after this overall transformation, a (special) unitary transformation just on the initial system alone does not matter. In the end we get $(d' - 1) - (d' - 1) + (d^2 d'^2 - 1) - (d^2 - 1) = d^2(d'^2 - 1)$ real parameters, which is the correct number for a quantum operation $d \rightarrow d'$.

3 The counterexample

3.1 The basic idea

We explain the basic idea behind the counterexamples in the simple setting of a map from a d -dimensional system to itself induced by a unitary map on the system together with a d' -dimensional environment. Imagine we initialize the d' -dimensional environment in a mixed state, say $\rho_{ini} = \rho|0\rangle\langle 0| + (1 - \rho)|1\rangle\langle 1|$. The resulting map (quantum operation) will be a probabilistic mixture (convex linear combination) of the two “pure-environment” maps we would get by initializing the environment either in state $|0\rangle$ or $|1\rangle$. A convex linear combination of quantum operations is defined through the convex linear combination of the density matrices to which they map. Also note that given one of these maps, the other has to fulfill certain conditions because they come from the same overall unitary transformation.

The main idea of our counterexample is that the fact that a pure state cannot be written as a non-trivial mixture of two pure states gives us strong restrictions on maps that map certain pure system states to pure states. For example, if a quantum operation maps $|0\rangle \rightarrow |0\rangle$, any “pure-environment” maps that make up part of a convex linear combination that gives this map have to map $|0\rangle$ to $|0\rangle$ as well.

3.2 The counterexample

As announced, we use the general framework of a quantum operation from an “initial” d -dimensional system to a “final” d' -dimensional system (Fig. 2). We will mark vectors in the final system with a prime: $|\cdot\rangle'$.

In our counterexample, we require that all but one of the initial system basis states go to one pure state, say $|0\rangle'$,

$$|i\rangle\langle i| \rightarrow |0\rangle'\langle 0|' \quad i = 0 \dots d - 2.$$

Before further specifying the counterexample, let us look at the consequences of this condition. If the target system has been initialized in a (truly, thus non-pure) mixed state, say $\rho'_{ini} = p|0\rangle'\langle 0|' + (1 - p)|1\rangle'\langle 1|'$ with $0 < p < 1$, we get

for the overall unitary transformation

$$|i\rangle|0\rangle' \rightarrow |\psi_i\rangle|0\rangle' \quad \text{and} \quad |i\rangle|1\rangle' \rightarrow |\xi_i\rangle|0\rangle' \quad i = 0 \dots d-2.$$

All $2(d-1)$ states on the right hand side have to be orthonormal, thus the $|\psi_i\rangle$ and $|\xi_i\rangle$ have to be orthonormal. But for $d > 2$ their number exceeds the dimension of the system. It is clear that were we to initialize the final system in a mixed state of rank > 2 , things would only be worse. It follows that the final system cannot start out in a mixed state, thus we assume it is initialized in the pure state $|0\rangle'$. (We treat the special case $d = 2$ below.)

Without loss of generality we set $|\psi_i\rangle = |i\rangle$. So we have $|i\rangle|0\rangle' \rightarrow |i\rangle|0\rangle'$ for $i = 0 \dots d-2$. The image of the remaining basis state $|d-1\rangle$ can be written

$$|d-1\rangle|0\rangle' \rightarrow \sum_{i=0}^{d-1} |i\rangle|\varphi_i\rangle'.$$

Now we require that the image of $|d-1\rangle$ be a truly mixed state ρ'_{d-1} . Furthermore, we require that under the quantum operation, $|d-1\rangle$ should “totally decohere” from the other basis states $|i\rangle$, meaning that a superposition should go to a mixture of the individual images of the states, so $(\alpha|i\rangle + \beta|d-1\rangle)(\bar{\alpha}\langle i| + \bar{\beta}\langle d-1|) \rightarrow |\alpha|^2|0\rangle'\langle 0|' + |\beta|^2\rho'_{d-1}$ ($i = 0 \dots d-2$). But the “totally decohere” condition means that $|\varphi_i\rangle' = 0$ for all $i \neq d-1$, from which it follows that the image of $|d-1\rangle$ would have to be pure after all.

3.3 The special case $d \rightarrow d'$ with $d = 2$

For $d = 2$ we still have to consider the case in which the final system is initialized in a mixed state, although from above it is clear that it would have to be a mixed state of rank 2, thus a mixture of just 2 different pure states. Thus $\rho'_{ini} = p|0\rangle'\langle 0|' + (1-p)|1\rangle'\langle 1|'$ with $0 < p < 1$. Then we have for the overall unitary transformation

$$|0\rangle|0\rangle' \rightarrow |0\rangle|0\rangle' \quad \text{and} \quad |0\rangle|1\rangle' \rightarrow |1\rangle|0\rangle'.$$

It follows that $|1\rangle|0\rangle' \rightarrow \alpha|0\rangle|0^\perp\rangle' + \beta|1\rangle|0'^\perp\rangle'$ where both $|0^\perp\rangle'$ and $|0'^\perp\rangle'$ are vectors orthogonal to $|0\rangle'$. Similarly for the image of $|1\rangle|1\rangle'$. We can now make our counterexample above work also for $d = 2$ by additionally requiring that the (truly mixed) image of $|d-1\rangle$ have some overlap with $|0\rangle'$, thus we require that

$$\langle 0|'\rho'_{d-1}|0\rangle' \neq 0.$$

3.4 Summary of the counterexample

In summary we give the counterexample map by its action on pure states:

$$\left(\sum_{i=0}^{d-2} \alpha_i |i\rangle + \beta |d-1\rangle \right) \left(\sum_{i=0}^{d-2} \bar{\alpha}_i \langle i| + \bar{\beta} \langle d-1| \right) \rightarrow \left(\sum_{i=0}^{d-2} |\alpha_i|^2 \right) |0\rangle'\langle 0|' + |\beta|^2 \rho'_{d-1},$$

where ρ'_{d-1} is not a pure state. For $d = 2$ we must additionally require that $\langle 0 | \rho'_{d-1} | 0 \rangle' \neq 0$. (Actually for $d' = 2$ this is always true.)

3.5 The counterexample is a quantum operation

Finally we show that the counterexample is a possible quantum operation, thus that with a large enough environment it can be implemented. So besides the initial and final systems we also have an environment. The map is simple, but for completeness we show it. The following overall unitary transformation implements the counterexample map (including the additional property required for the case $d = 2$), with $\alpha, \beta \neq 0$:

$$\begin{aligned} |i\rangle|0\rangle'|0\rangle_{env} &\rightarrow |i\rangle|0\rangle'|0\rangle_{env} & i = 0 \dots d-2 \\ |d-1\rangle|0\rangle'|0\rangle_{env} &\rightarrow \alpha|d-1\rangle|0\rangle'|0\rangle_{env} + \beta|d-1\rangle|1\rangle'|1\rangle_{env} \end{aligned}$$

4 Further remarks

4.1 How many counterexamples are there?

Once a single counterexample has been found, basic topological properties of the (convex) set of all quantum operations and the (also convex) subset of those that can be realized with a d -dimensional (possibly) mixed environment imply that the set of counterexamples contains balls and thus is a set of non-zero measure. All that is needed is that the subset of maps that that can be realized with a d -dimensional mixed environment, being convex, is closed, and that the “outer” set is a regular set, thus it is a closure of its inner points. The latter is simply because the set of all quantum operations is a “voluminous” convex set, thus one which does not happen to lie in some hyperplane.

Note that our counterexample lies on the boundary of the set of quantum operations, so we must take the intersection of a ball around it with the set of quantum operations to find inner points of the set of counterexamples. Around these points there will then be balls of counterexamples.

Also note that the counterexample given in [1], for the single qubit case ($d = 2$), is unital and thus very different from ours.

A complete characterization of exactly which maps can be realized with a d -dimensional mixed environment and which cannot has yet to be found even in the single qubit case. Ruskai et al. [6] give a parameterization of the space of all quantum operations for the single qubit space that might be useful here.

4.2 How big an environment is necessary?

From the counterexamples for $d > 2$ we can see that more could be said about how large a mixed environment has to be, but we haven't investigated further.

4.3 Implications for simulating quantum operations (?)

It looks like our subject is mostly a mathematical challenge. In [1] it was said that one may want to simulate quantum operations (e.g. on a quantum computer) with as few resources as possible (see also [5]). But Choi [4] shows that every extremal map of the convex set of quantum operations can be realized with a d -dimensional environment initialized in a pure state. To achieve any mixture of such extremal maps (and thus any map) one can simply carry out each of them with a certain probability, using a probabilistic protocol. On the other hand certain extremal maps do need an environment of dimension d , thus this probably is a necessary resource.

4.4 What about “Markovian” quantum operations?

By “Markovian” quantum operations we mean those quantum operations that can be “generated” from ones that are infinitesimally close to the identity. Of course this notion makes sense only for dimension preserving quantum operations. It is known that not all quantum operations can be generated in this way, so we may wonder whether all those maps could be implemented with a “small” mixed environment.

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